

# Wave Propagation in a Viscoelastic Solid

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A study is made of the propagation of waves in a viscoelastic solid from a Heaviside step function plane source (plane waves) and from a Heaviside step function point source (spherical waves). The work is based on solutions of a normalized form of the Stokes wave equation, and the displacements are expressed in terms of finite integrals which are readily evaluated by numerical quadrature. A comparison is made with the work of Ricker, and with earlier work of the present author.

## 1. INTRODUCTION

In a recent paper the author [1] has shown how Ricker's generating, displacement, velocity, and acceleration functions pertaining to the propagation of a pulse in a viscoelastic medium [2] can be transformed into finite integrals which are easily evaluated by numerical quadrature. The present paper extends these results to cover a Heaviside step function-type source and makes a comparison with earlier results due to the author [3, 4] for this same type of source. The treatment is based on a normalized form of the Stokes wave equation, and the plane wave propagation from a plane source is treated simultaneously with the spherical wave propagation from a point source. An interesting result concerning the shape of the displacement-time graph  $\psi(x, t)$  at any given distance  $x$  from the plane source is presented. A time integral and various derivatives of  $\psi(x, t)$  are expressed as finite integrals, and in fact the derivation of these integral representations is the main purpose of this paper.

## 2. THEORY

Ricker [2] has shown that the function

$$\psi_1 = e^{-\alpha x} \cos \beta(t - x/v) \tag{1}$$

satisfies the normalized Stokes wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 \psi}{\partial x^2 \partial t} = \frac{\partial^2 \psi}{\partial t^2} \tag{2}$$

if

$$\begin{aligned}\beta &= \tan 2\phi, \\ \alpha &= \tan 2\phi(\cos 2\phi)^{1/2} \sin \phi, \\ \beta/v &= \tan 2\phi(\cos 2\phi)^{1/2} \cos \phi.\end{aligned}\tag{3}$$

Here  $t$  denotes numerical (dimensionless) time, and  $x$  denotes numerical distance from a plane [1, 2], while  $\phi$  ( $0 \leq \phi < \pi/4$ ) is a convenient parametric representation for the numerical angular frequency  $\beta$ .

By differentiating (1) with respect to  $t$ , we also readily obtain the solution

$$\psi_2 = e^{-\alpha x} \sin \beta(t - x/v),\tag{4}$$

with the same values of  $\alpha$  and  $\beta/v$ .

As observed by Ricker [2] we may treat the problem of plane wave propagation from a plane source, and the spherical wave propagation from a point source simultaneously. This is done by noting that the normalized Stokes wave equation for the latter problem has the form

$$\frac{\partial^2}{\partial R^2} \left[ R\Psi + \frac{\partial}{\partial t} (R\Psi) \right] = \frac{\partial^2}{\partial t^2} (R\Psi),\tag{5}$$

where  $R$  denotes the radial distance from the source. Clearly, solutions of (5) may be obtained from solutions of (2) by replacing  $x$  by  $R$  and dividing by  $R$ ; or in symbols

$$\Psi(R, t) = \psi(R, t)/R.\tag{6}$$

In this paper we propose merely to treat the plane wave problem, and it is evident from (6) how to obtain solutions to the spherical wave problem, the same transformation being applied, of course, to the source.

Going back to Eqs. (1)–(4), we assume that  $\psi$ ,  $\psi_1$ , or  $\psi_2$  represents particle displacement in the positive  $x$  direction at numerical time  $t$  for any particle in the plane at numerical distance  $x$  from the source plane  $x = 0$ .

By setting  $x = 0$  it is evident that our solutions (1) and (4) of (2) become

$$\psi_1 = \cos \beta t\tag{7}$$

and

$$\psi_2 = \sin \beta t,\tag{8}$$

respectively, in the plane  $x = 0$ , and we can represent our actual source by superposition of such elementary solutions.

We now suppose that our initial disturbance in the source plane  $x = 0$  is  $H(t)$ , the Heaviside unit function defined by

$$\begin{aligned}H(t) &= 0, & t < 0, \\ &= 1, & t > 0.\end{aligned}\tag{9}$$

In order to solve (2) for this type of source, it is convenient to represent  $H(t)$  in terms of our basic functions ((7), (8)). We readily find that

$$H(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin \beta t}{\beta} d\beta. \tag{10}$$

It is also convenient to use the representation

$$H(t) = \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \frac{e^{i\beta t}}{i\beta} d\beta, \tag{11}$$

where the symbol  $0^+$  for the lower limit means that the path of integration includes a quarter circle, of radius  $\epsilon$  tending to zero, around the origin in the clockwise (positive) sense (Fig. 1), and  $\operatorname{Re}$  denotes the real part. This notation will be used throughout this paper even if (as in the present case) the imaginary part of the integral is infinite.

We now immediately find, from (10), (8), and (4), the displacement  $\psi(x, t)$  for waves propagating from a Heaviside step function source  $H(t)$  in the plane  $x = 0$ :

$$\psi(x, t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} e^{-\alpha x} \sin \beta (t - x/v) \frac{d\beta}{\beta}, \tag{12}$$

where  $\alpha, \beta/v$  are given as functions of  $\beta$  (Eq. (3)). This result may be written in the form

$$\begin{aligned} \psi(x, t) = & \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \exp \left[ -x\beta(1 + \beta^2)^{-1/4} \sin \left( \frac{\tan^{-1} \beta}{2} \right) \right] \\ & \cdot \sin \left[ \beta t - x\beta(1 + \beta^2)^{-1/4} \cos \left( \frac{\tan^{-1} \beta}{2} \right) \right] \frac{d\beta}{\beta}. \end{aligned} \tag{13}$$

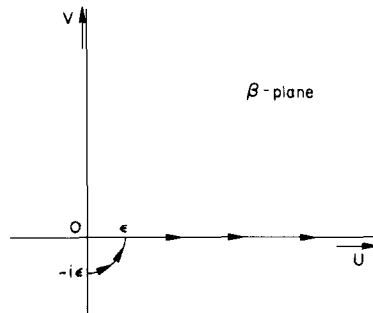


FIG. 1. Path of integration for  $\int_0^{\infty}$ . The small quarter circle has radius  $\epsilon$ , and  $\epsilon \rightarrow 0$ .

A direct comparison with Ricker's generating function  $\Phi$  (for Ricker's plane wave case) is obtained by differentiating (13) with respect to  $t$ , and we find that

$$\frac{\partial \psi}{\partial t}(x, t) = \frac{1}{2\pi} \Phi(x, t). \quad (14)$$

It is clear, however, that (13) is not convenient for computations, in view of the oscillatory nature of the integrand and the infinite range of integration. Accordingly, we transform the path of integration in the  $\beta$  plane. For this purpose it is more convenient to use the representation (11) for  $H(t)$ , and then in place of (13) we obtain the more compact form

$$\psi(x, t) = \frac{1}{\pi} \operatorname{Re} \int_{0^+}^{\infty} \exp \left[ \frac{-i\beta x}{(1+i\beta)^{1/2}} + i\beta t \right] \frac{d\beta}{i\beta}, \quad (15)$$

the path of integration being as in Fig. 1.

In the earlier paper on the calculation of Ricker seismic wavelet functions, the author [1] has shown how to transform the path of integration in the  $\beta$  plane so that the exponent

$$\frac{-i\beta x}{(1+i\beta)^{1/2}} + i\beta t \quad (16)$$

becomes real on the new path. The transformation was then used to obtain finite integral representations for Ricker's wavelet functions. In the present paper the same transformation is appropriate for our equation (15) for  $\psi(x, t)$ . For convenience we describe this transformation briefly, but for full details the reader is referred to the earlier paper [1].

We define a parameter

$$k = x/t \quad (17)$$

and distinguish three cases:  $k < 1$ ,  $k > 1$ ,  $k = 1$ . In the complex  $\beta$  plane, we define  $r, \theta$  by

$$1 + i\beta = r e^{\theta i}, \quad (18)$$

so that

$$(1 + i\beta)^{1/2} = r^{1/2} e^{\theta i/2} \quad (19)$$

(we cut the  $\beta$  plane along a line through the branch point  $\beta = i$ , and parallel to the negative real axis), and we also write

$$\beta = u + iv, \quad (20)$$

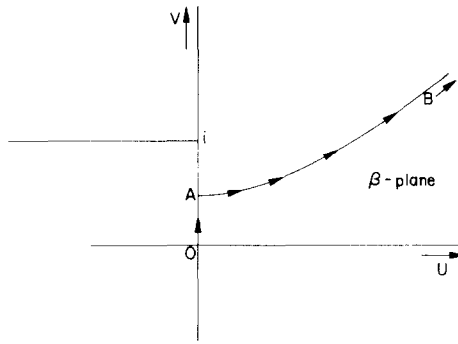


FIG. 2. Schematic diagram showing transformed path of integration  $OAB$  for the case  $k < 1$ . The  $\beta$ -plane is cut through the branch point  $\beta = i$ , parallel to the negative real axis.

A path is then chosen so that the exponent (16) becomes real. It is found that this occurs when

$$\theta = 0 \tag{21}$$

or when

$$r^{-1/2} + r^{-3/2} = (2/k) \cos \theta/2, \tag{22}$$

which is indicated as the curved path  $AB$  in Fig. 2 for the case  $k < 1$  and starts on the imaginary axis in the  $\beta$  plane at the point

$$A \equiv (0, 1 - r_0), \tag{23}$$

where  $r_0$  is the positive root of the equation,

$$r^{-1/2} + r^{-3/2} = 2/k, \tag{24}$$

obtained from (22) on putting  $\theta = 0$ . The path of integration chosen goes from the origin  $O$  up or down the imaginary axis to the point  $A$ , and thence along the curve  $AB$  to infinity (Fig. 2). Since it is the real part of the integral that is required in (15), there is no contribution from the straight portion  $OA$  of the path, since on  $OA$ ,

$$\beta = iv, \quad d\beta = idv$$

and

$$\frac{d\beta}{i\beta} = -\frac{idv}{v},$$

which is purely imaginary. Also on the curved path  $AB$  the value of the exponent (16) is

$$p = (r + 1)(k^2 r^{-2} - 1)t. \tag{25}$$

We need to take due account of the singularity at 0, and when we do this, we find

$$\psi(x, t) = \frac{1}{\pi} \int_0^\pi e^{\rho} \frac{r^3 - k^2}{(r+3)(r^2 - k^2)} d\theta \quad (26)$$

for  $k > 1$ . If  $k < 1$  we have to add 1 to this result, while for  $k = 1$  we have to add  $\frac{1}{2}$  to the result (26). It should be emphasized that there is, nevertheless, *no* discontinuity in  $\psi(x, t)$  at  $t = x$  (i.e.,  $k = 1$ ), and this is also borne out by the numerical computations. Here the factor

$$\frac{r^3 - k^2}{(r+3)(r^2 - k^2)} d\theta$$

is obtained as the value of

$$\operatorname{Re} \left( \frac{d\beta}{i\beta} \right)$$

on the path (22), after a simple, but somewhat lengthy, algebraic calculation. Also, the integration variable has been taken as  $\theta$ , so that we have a finite range of integration. It should be noted also that  $p \rightarrow -\infty$  quite rapidly as  $\theta \rightarrow \pi$  ( $t > 0$ ), and the integral (26) converges for all (positive) values of the parameters  $t, k, r$  has to be obtained as a function of  $\theta$  from (22), which we write in the form

$$r^{-1/2} + r^{-3/2} = 2\gamma, \quad (27)$$

where

$$\gamma = (1/k) \cos \theta/2. \quad (28)$$

Finding  $r$  involves solving a cubic equation, but this is done readily, and we have explicitly

$$r = 3/(\rho^{1/3} - \rho^{-1/3})^2, \quad (29)$$

where

$$\rho = 3 \sqrt{3} \gamma + (1 + 27\gamma^2)^{1/2}. \quad (30)$$

Incidentally, it is not difficult to show that the integral

$$J = \frac{1}{\pi} \int_0^\pi \frac{r^3 - k^2}{(r+3)(r^2 - k^2)} d\theta \quad (31)$$

has the values 0,  $\frac{1}{2}$ , 1 for  $k < 1$ ,  $k = 1$ ,  $k > 1$ , respectively.

By differentiations of (15) and application of our transformation, we obtain the following results, after some algebraic manipulations:

$$\frac{\partial \psi}{\partial t} = \frac{1}{\pi} \int_0^\pi e^{\rho} \frac{r^4 - r^3 + k^2(r+1)^2}{r^2(r+3)} d\theta \quad (32)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{\pi} \int_0^\pi e^p \frac{r+1}{r^4(r+3)} [r^5 - 3r^6 + k^2 r^2 (r+1)(r^2 - 3r - 1) + k^4 (r+1)^3] d\theta \quad (33)$$

$$\frac{\partial \psi}{\partial x} = -\frac{1}{\pi} \int_0^\pi e^p \frac{2r^3 + k^2(r+1)}{kr(r+3)} d\theta \quad (34)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\pi} \int_0^\pi e^p \frac{r+1}{2r^2(r+3)} [2r^3 - 6r^2 + 2k^2(r-1)] d\theta \quad (35)$$

$$\frac{\partial^3 \psi}{\partial x^2 \partial t} = -\frac{1}{\pi} \int_0^\pi e^p \frac{(r+1)^2}{2r(r+3)} [6r^5 - 6r^4 + 2k^2 r^2 (-r^2 + 3r + 2) - 2k^4 (r+1)^2] d\theta \quad (36)$$

where  $p$  is given by (25). From the integrands of (33), (35), (36) we can (incidentally) verify immediately that  $\psi$  indeed satisfies the Stokes wave equation (2). Also, (32) has been given in a slightly different, but equivalent, form in the author's earlier paper [1] as Ricker's generating function  $\Phi$ .

In the next section we establish an interesting property of the shape of the displacement-time graph  $\psi(x, t)$  as a function of  $t$  for given  $x$ , and incidentally extend our results by giving a finite integral representation for  $\int_0^t \psi(x, \tau) d\tau$ .

### 3. THE DISPLACEMENT-TIME RECORD

In Section 4 the results of the calculation of  $\psi(x, t)$  as a function of  $t$  are tabulated and plotted for two values of  $x$ . In Fig. 3, for instance,  $\psi(1, t)$  is plotted against  $t$  and compared with the function  $H(t-1)$ , which would represent the displacement in the case of ordinary elasticity in which (2) is replaced by the ordinary wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial t^2}, \quad (37)$$

assuming the same Heaviside unit function source in the plane  $x=0$ . In general, the situation is as depicted schematically in Fig. 5 (the comparison being with the function  $H(t-x)$ ), and we propose to establish here that the shaded areas are equal, i.e., that

$$\int_0^x \psi(x, t) dt = \int_x^\infty [1 - \psi(x, t)] dt, \quad (38)$$

which means that (in this sense) the center of the pulse travels with unit numerical velocity. Equation (38) may be written in the equivalent form

$$\int_0^\infty [1 - \psi(x, t)] dt = x, \quad (39)$$

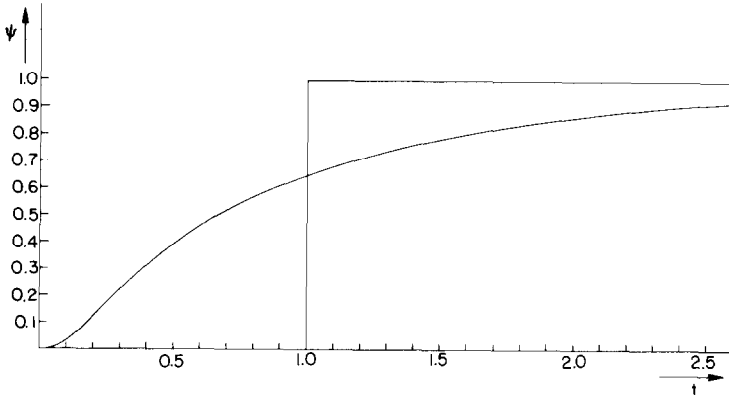


FIG. 3. Displacement-time record  $\psi(1, t)$  compared with the step function  $H(t - 1)$ .

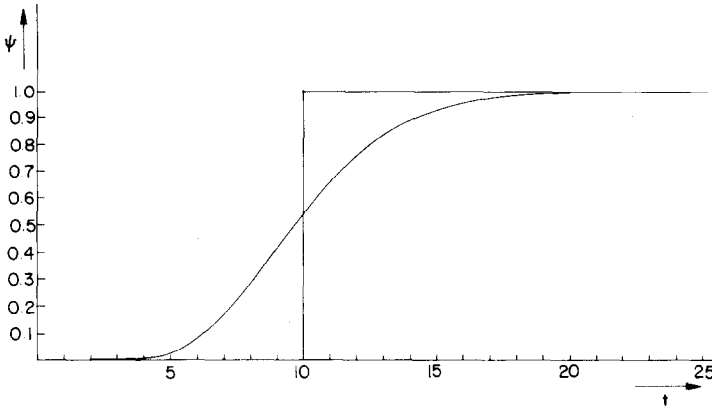


FIG. 4. Displacement-time record  $\psi(10, t)$  compared with the step function  $H(t - 10)$ .

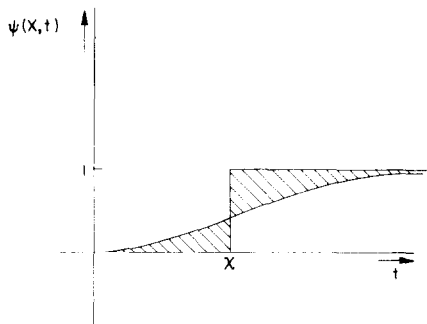


FIG. 5. Schematic displacement-time record  $\psi(x, t)$  compared with the step function  $H(t - x)$ . The shaded areas are equal.



and in order to establish this we seek a representation for

$$\int_0^t \psi(x, \tau) d\tau$$

similar to Eqs. (26) and (32)–(36).

The analysis is a little complicated, but basically we start again from (15) and integrate with respect to  $t$ , giving

$$\int_0^t \psi(x, \tau) d\tau = -\frac{1}{\pi} \operatorname{Re} \left[ \int_{0^+}^{\infty} \exp \left\{ \frac{i\beta x}{(1+i\beta)^{1/2}} + i\beta\tau \right\} \frac{d\beta}{\beta^2} \right]_{\tau=0}^t.$$

which may be shown to give

$$\int_0^t \psi(x, \tau) d\tau = -\frac{1}{\pi} \operatorname{Re} \int_{0^+}^{\infty} \exp \left[ \frac{-i\beta x}{(1+i\beta)^{1/2}} + i\beta\tau \right] \frac{d\beta}{\beta^2}. \tag{40}$$

We now apply our transformation as before, taking due note that since we now have a double pole at the origin, integrals on *incomplete* small circles around the origin have to be handled with great care. The final result may be put in the form

$$\int_0^t \psi(x, \tau) d\tau = t - x - \frac{1}{\pi} \int_{\theta=0}^{\pi} e^{\rho} \frac{d\beta}{\beta^2}, \tag{41}$$

for  $t > x$ , i.e.,  $k < 1$ , while the term  $t - x$  must be omitted for  $t < x$ , i.e.,  $k > 1$ . An elementary calculation shows that on our new integration path

$$-\operatorname{Re} \left( \frac{d\beta}{\beta^2} \right) = \frac{k^2 r^2 (r^3 + 2r^2 + 1) - r^5 - 3r^6}{(r+3)(r+1)^2 (k^2 - r^2)^2} d\theta, \tag{42}$$

and this expression is to be substituted into (41).

In the case  $t = x$ , i.e.,  $k = 1$ , the calculations are somewhat different, and we obtain

$$\int_0^x \psi(x, \tau) d\tau = \frac{1}{\pi} \int_0^{\pi} \frac{e^{\rho} - 1}{(1-r^2)} \frac{r^2 (3r^2 + 1)}{(r+3)(r+1)^2} d\theta. \tag{43}$$

Here the limiting value of the integrand when  $\theta \rightarrow 0$  is  $x/2$ .

The desired result (39) follows immediately from (41) by letting  $t \rightarrow \infty$ .

#### 4. NUMERICAL RESULTS

The displacement  $\psi(x, t)$  has been calculated as a function of  $t$  for  $x = 1$  and  $x = 10$ . The results were obtained in the computer using gaussian quadrature applied to the integral (36) and are tabulated in Tables I and II, where they are compared with results obtained in earlier papers by the author [3, 4] using approximate Laplace

TABLE I  
 Values of  $\psi(x, t)$  for  $x = 1$ ,  $t = 0(0.1)$ , 3.0 (0.5), 10.0<sup>a</sup>

$t$	$\psi(1, t)$	$g_1(t)$
0	0	0
0.1	0.0273611	(0.0304)
0.2	0.1295178	(0.1282)
0.3	0.2329710	0.2328
0.4	0.3223330	0.3224
0.5	0.3983864	0.3985
0.6	0.4635544	0.4636
0.7	0.519934	0.5199
0.8	0.56914	0.5691
0.9	0.6124	0.6124
1.0	0.6506316	0.65063
1.1	0.68466	0.68461
1.2	0.71492	0.71491
1.3	0.742031	0.74203
1.4	0.7663547	0.766354
1.5	0.7882223	0.78822
1.6	0.8079171	0.807917
1.7	0.8256824	0.8256683
1.8	0.8417278	0.841728
1.9	0.8562658	0.856236
2.0	0.8693663	0.869367
2.1	0.8812596	0.881260
2.2	0.8920401	0.892040
2.3	0.9018177	0.901818
2.4	0.9106906	0.910691
2.5	0.9187463	0.918746
2.6	0.9260630	0.926063
2.7	0.9327109	0.932711
2.8	0.9387531	0.938753
2.9	0.9442463	0.944246
3.0	0.9492417	0.949242
3.5	0.9682182	
4.0	0.9800759	
4.5	0.9875021	
5.0	0.9921589	
5.5	0.9950807	
6.0	0.9969144	
6.5	0.9980651	
7.0	0.9987872	
7.5	0.9992400	
8.0	0.9995240	
8.5	0.9997020	
9.0	0.9998135	
9.5	0.9998834	
10.0	0.9999271	

<sup>a</sup> Also, for comparison, values of the equivalent quantity  $g_o(t, x)$  for  $x = 1$ ,  $\sigma = 1/x = 1$  taken from a previous paper by the author [4] are given. Values in parentheses are only approximate.

TABLE II  
 Values of  $\psi(x, t)$  for  $x = 10$ ,  $t = 0(1), 20^a$

$t$	$\psi(10, t)$	$g_{0.1}(t/10)$
0	0	0
1	$3.905 \times 10^{-12}$	(0.0087)
2	$3.055 \times 10^{-6}$	(-0.0023)
3	$4.099 \times 10^{-4}$	(-0.0006)
4	$5.500 \times 10^{-3}$	(0.0062)
5	0.0273975	0.0274
6	0.0800106	0.0800
7	0.1685325	0.1685
8	0.28597	0.2860
9	0.4175	0.4175
10	0.5474849	0.54748
11	0.6639	0.66384
12	0.76020	0.76020
13	0.83507	0.835068
14	0.8902114	0.890211
15	0.9290335	0.929034
16	0.9553255	0.955326
17	0.9725408	0.972541
18	0.9834838	0.983484
19	0.9902597	0.9902597
20	0.9943582	0.9943582

<sup>a</sup> Also, for comparison, values of the equivalent quantity  $g_{\sigma}(t, x)$  for  $x = 10$ ,  $\sigma = 1/x = 0.1$  taken from a previous paper by the author [4] are given. Values in parentheses are only approximate.

TABLE III  
 Values of  $\int_0^t \psi(1, \tau) d\tau$  for several values of  $t^a$

$t$	$\int_0^t \psi(1, \tau) d\tau$	
	a	b
0.5	0.0906	0.0911
0.9	0.3018	0.2969
1.0	0.3597	0.3601
2.0	1.1386	1.1388

<sup>a</sup> Column a is obtained from Eqs. (41) through (43) and may be compared with column b obtained by very rough numerical quadrature from Table I.

transform inversion techniques based on rational approximations. The results are also displayed in graphical form in Figs. 3 and 4. As a check on the accuracy, the numerical integrations were carried out using both 16 and 32 gaussian points, and the results are given to seven places of decimals, except in the vicinity of  $t = x$ , where less accuracy was obtained.

All results in Tables I and II are believed to be accurate to the number of places shown, except for possible slight inaccuracies in  $g_\sigma(t/x)$ , and except for the values in parentheses as noted. The important point here is that the method of the present paper is essentially simpler than those in the author's previous papers, in that no numerical Laplace transform inversion has to be performed. Also the results (32)–(36) and (41)–(43) show that the method is immediately applicable to the calculation of various derivatives and integrals of  $\psi(x, t)$  and in general should be useful for further research on viscoelastic wave propagation.

Table III gives a few values for  $\int_0^t \psi(1, \tau) d\tau$  for some selected values of  $t$ , and a comparison is made with results obtained by very approximate (trapezium rule) numerical quadrature from Table I.

## 5. CONCLUSION

A method of calculation due to the author has been used to obtain finite integral representations for the displacement and related quantities in a homogeneous viscoelastic solid for a plane wave emanating from a step function plane source. Examples of numerical calculations based on these integrals have been given, high accuracy being readily attainable. It is hoped that the method will facilitate further research in this field.

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